

# ON HADAMARD PRODUCTS OF LINEAR VARIETIES

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**ABSTRACT.** In this paper we address the Hadamard product of linear varieties not necessarily in general position.

In  $\mathbb{P}^2$  we obtain a complete description of the possible outcomes. In particular, in the case of two disjoint finite sets  $X$  and  $X'$  of collinear points, we get conditions for  $X \star X'$  to be either a collinear finite set of points or a grid of  $|X||X'|$  points.

In  $\mathbb{P}^3$ , under suitable conditions (which we prove to be generic), we show that  $X \star X'$  consists of  $|X||X'|$  points on the two different rulings of a non-degenerate quadric and we compute its Hilbert function in the case  $|X| = |X'|$ .

## 1. INTRODUCTION

The Hadamard product of matrices is well known in linear algebra: it has nice properties in matrix analysis ([HM, Liu, LT]) and has applications in both statistics and physics ([Liu2, LN, LNP]). Recently, in the papers [CMS, CTY], the authors define a Hadamard product between projective varieties  $X, Y \subset \mathbb{P}^n$ , denoted  $X \star Y$ , as the closure of the image of the rational map

$$X \times Y \dashrightarrow \mathbb{P}^n, \quad ([a_0 : \cdots : a_n], [b_0 : \cdots : b_n]) \mapsto [a_0 b_0 : a_1 b_1 : \cdots : a_n b_n].$$

and use it to describe the algebraic variety associated to the restricted Boltzmann machine, which is the undirected graphical model for binary random variables specified by the bipartite graph  $K_{r,n}$  (note that [CTY] concerns the case  $r = 2, n = 4$ ). Using the definition in [CMS], the first author, together with E. Carlini and J. Kileel, started to study more deeply the Hadamard product of projective varieties, with particular emphasis for the case of linear spaces ([BCK]).

Since the Hadamard product of varieties is far to be completely understood and studied, our paper wants to be a natural continuation of the paper [BCK]. Here we still study the case of both linear spaces and zero-dimensional schemes in  $\mathbb{P}^2$  and  $\mathbb{P}^3$ , dropping the hypothesis that these varieties be in general position. The condition to be not in general position means that points can have many zero coordinates and linear spaces can intersect coordinate hyperplanes in dimension greater than the expected one for the case of general position. This fact forces us to study all possible pathological behaviors that then can happen in the Hadamard product of such varieties. For the case of  $\mathbb{P}^2$ , Theorem

3.4 gives a complete classification of all possible cases of the Hadamard product between a point and a line, while Theorem 3.6 studies all possible cases of incidence of  $Q \star L$  and  $Q' \star L$  for two distinct points  $Q, Q'$  and a line  $L$ . These results lead to Theorem 3.12 where we prove that, under suitable conditions, the Hadamard product of two sets of collinear points  $X$  and  $Y$  is a complete intersection. Recall that a set  $X$  of points of  $\mathbb{P}^n$  is a *complete intersection* if its ideal  $I_X$  is generated by  $n$  forms.

Turning to the case of  $\mathbb{P}^3$ , we notice that the Hadamard product of two sets of collinear points  $X$  and  $Y$  is not, in general, a complete intersection. However, we can prove, under generic assumptions, that  $X \star Y$  is a grid on a quadric (Theorem 4.2) and we are able to compute its Hilbert function when  $|X| = |Y|$  (Theorem 4.8). In this case we also prove that  $X \star Y$  is never a complete intersection (assuming  $|X| = |Y| > 1$ ).

We work over an algebraically closed field  $\mathbb{K}$ .

We denote by  $HF_X$  the *Hilbert function* of a finite set of projective points  $X$ , that is  $HF_X(t) = \dim_{\mathbb{K}} R/I_{X,t}$ , where  $R = \mathbb{K}[x_0, \dots, x_n]$  and  $I_X$  is the (radical) ideal defining  $X$ . It is well known that, after being strictly increasing,  $HF_X(t) = |X|$  for  $t$  sufficiently large and we call *regularity index* of  $X$ , denoted by  $\tau_X$ , the least degree where this happens:

$$\tau_X = \min\{t \geq 0 : HF_X(t) = |X|\}.$$

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## 2. GENERAL RESULTS IN $\mathbb{P}^n$

As in [BCK],  $H_i \subset \mathbb{P}^n$  denotes the hyperplane defined by  $x_i = 0$  and

$$\Delta_i = \bigcup_{0 \leq j_1 < \dots < j_{n-i} \leq n} H_{j_1} \cap \dots \cap H_{j_{n-i}}.$$

Recall that  $\Delta_i$  can be viewed as the  $i$ -dimensional variety of points having at most  $i + 1$  non-zero coordinate, equivalently at least  $n - i$  zero coordinates.

We set  $\Delta_{-1}$  to be the set  $\{(0, \dots, 0)\}$  and we write  $P \star Q \in \Delta_{-1}$  if it is not defined.

It easily follows from Lemma 3.2 of [BCK] that:

### Theorem 2.1.

- (1) Let  $P, Q, A$  be points of  $\mathbb{P}^n$  with  $A \notin \Delta_{n-1}$ , then  $P \star A = Q \star A$  if and only if  $P = Q$ .
- (2) Let  $H \subset \mathbb{P}^n$  be a hyperplane defined by  $a_0x_0 + \dots + a_nx_n = 0$  and such that  $H \cap \Delta_0 = \emptyset$ , let  $P, Q$  be points not in  $\Delta_{n-1}$  with

- $P = [p_0 : \cdots : p_n]$ , then  $P \star H : \{\frac{a_0}{p_0}x_0 + \cdots + \frac{a_n}{p_n}x_n = 0\}$  and  $P \star H = Q \star H$  if and only if  $P = Q$ .
- (3) Let  $P \notin \Delta_{n-1}$ , let  $H, K$  be two hyperplanes such that  $H \cap \Delta_0 = \emptyset = K \cap \Delta_0$ , then  $P \star H = P \star K$  if and only if  $H = K$ .

**Theorem 2.2.** Let  $P \in \mathbb{P}^n \setminus \Delta_{n-1}$ , let  $H, K$  be two hyperplanes such that  $H \cap \Delta_0 = \emptyset = K \cap \Delta_0$ , then  $P \star (H \cap K) = (P \star H) \cap (P \star K)$ .

*Proof.* We may assume  $H \neq K$ , for otherwise the result is trivial.

For all  $Q \in H \cap K$ , we have  $P \star Q \in P \star (H \cap K)$  and  $P \star Q \in (P \star H) \cap (P \star K)$ , hence  $P \star (H \cap K) \subseteq (P \star H) \cap (P \star K)$ , being the right-hand side a closed set. To see the other inclusion, by (3) of Theorem 2.1, we have  $P \star H \neq P \star K$ , then  $(P \star H) \cap (P \star K)$  is a linear subspace of dimension  $n - 2$ . Since  $P \notin \Delta_{n-1}$ , it follows from Lemma 3.1 of [BCK] that  $P \star (H \cap K)$  is a linear subspace of dimension  $n - 2$ . Therefore  $P \star (H \cap K)$  is the intersection of the hyperplanes  $P \star H$  and  $P \star K$ .  $\square$

**Corollary 2.3.** Let  $P \in \mathbb{P}^3 \setminus \Delta_2$  and let  $H, K$  be two planes such that  $L = H \cap K$  and  $H \cap \Delta_0 = \emptyset = K \cap \Delta_0$ , then  $P \star L$  is the intersection of the two planes  $P \star H$  and  $P \star K$ .

Now we look at the products of two hyperplanes  $H$  and  $K$ .

**Remark 2.4.** If  $H$  and  $K$  are coordinate hyperplanes respectively defined by  $x_i = 0$  and  $x_j = 0$ , then  $H \star K$  is the hyperplane defined by  $x_i = 0$ , when  $i = j$  and the linear subspace defined by  $x_i = x_j = 0$ , when  $i \neq j$ .

**Theorem 2.5.** Let  $H, K$  be the hyperplanes of  $\mathbb{P}^n$  defined by  $a_i x_i + a_j x_j = 0$  and  $b_i x_i + b_j x_j = 0$  respectively, with  $i \neq j$  in  $\{0, \dots, n\}$  and either  $a_i a_j \neq 0$  or  $b_i b_j \neq 0$ . Then  $H \star K$  is the hyperplane defined by  $a_i b_i x_i - a_j b_j x_j = 0$ .

*Proof.* For simplicity of notation we may assume  $i = 0$  and  $j = 1$ , the other case being similar.

We distinguish the following two cases:

- (1) If  $a_1 = 0$  and  $b_0 b_1 \neq 0$ , then  $H : \{x_0 = 0\}$ . Let  $P = [0 : p_1 : p_2 : \cdots : p_n] \in H$  and let  $Q \in K$ , then  $Q = [-\frac{b_1}{b_0}q_1 : q_1 : q_2 : \cdots : q_n]$ , and so  $P \star Q = [0 : p_1 q_1 : \cdots : p_n q_n]$ , i.e.  $H \star K : \{x_0 = 0\}$ .
- (2) If  $a_0 a_1 \neq 0$  and  $b_0 b_1 \neq 0$ , then  $P = [-\frac{a_1}{a_0}p_1 : p_1 : p_2 : \cdots : p_n] \in H$  and  $Q = [-\frac{b_1}{b_0}q_1 : q_1 : q_2 : \cdots : q_n] \in K$ , thus  $P \star Q = [\frac{a_1 b_1}{a_0 b_0}p_1 q_1 : p_1 q_1 : p_2 q_2 : \cdots : p_n q_n]$ . We claim that  $H \star K$  is the hyperplane  $L : \{a_0 b_0 x_0 - a_1 b_1 x_1 = 0\}$ . It is obvious that  $H \star K \subseteq L$ . To see the other inclusion let  $S = [s_0 : \cdots : s_n] \in L$  and let  $P = [-\frac{a_1}{a_0} : 1 : 1 : \cdots : 1] \in H$  and consider  $Q = [-\frac{a_0 s_0}{a_1} : s_1 : s_2 : \cdots : s_n]$ . Clearly  $P \star Q = S$  and  $b_0 \frac{-(a_0 s_0)}{a_1} + b_1 s_1 = 0$ , i.e.  $Q \in K$ .  $\square$

**Corollary 2.6.** *Let  $H \subset \mathbb{P}^n$  be the hyperplane defined by  $a_i x_i + a_j x_j = 0$ , with  $i \neq j$  in  $\{0, \dots, n\}$  and  $a_i a_j \neq 0$ , then  $H \star H$  is the hyperplane defined by  $a_i^2 x_i - a_j^2 x_j = 0$ .*

*Proof.* It follows immediately from Theorem 2.5 with  $H = K$ .  $\square$

**Remark 2.7.** If  $Q \in H_i$ , for some  $i \in \{0, \dots, n\}$ , then, for all  $X \subseteq \mathbb{P}^n$ , we have  $Q \star X \subseteq H_i$ , hence  $H_i \star X \subseteq H_i$ .

**Example 2.8.** Let  $H$  and  $K$  be the planes in  $\mathbb{P}^3$  of equations respectively

$$H : 3x_1 - 2x_3 = 0 \quad K : -7x_1 + 4x_3 = 0.$$

Using the procedure, in **Singular**, described in Section 5 we get

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ring r=0,(x(0..3)),dp;
ideal H=3*x(1)-2*x(3);
ideal K=-7*x(1)+4*x(3);
HPr(H,K,3);
_[1]=21*x(1)-8*x(3)
and, in particular,
HPr(H,H,3);
_[1]=9*x(1)-4*x(3)
```

### 3. SETS OF COLLINEAR POINTS IN $\mathbb{P}^2$

Now we focus on  $n = 2$  and on sets of at least two collinear points.

The following Corollary is an application of the results of the previous section.

**Corollary 3.1.** *Let  $X, Y$  be two sets of collinear points in  $\mathbb{P}^2$  such that  $X \cup Y$  is contained in a line  $L$  such that  $L \cap \Delta_0 \neq \emptyset$ , then  $X \star Y$  are collinear points contained in the line  $L \star L$ .*

We need the following technical Lemma, whose proof follows easily from the Definitions:

**Lemma 3.2.** *Let  $L$  be a line in  $\mathbb{P}^2$  such that  $L \cap \Delta_0 = \emptyset$ . Then*

- (1)  $|L \cap \Delta_1| = 3$  and  $|L \cap \Delta_1 \cap H_i| = 1$ , for all  $i \in \{0, 1, 2\}$ .
- (2) For all  $P, Q \in L \cap \Delta_1$ , we have that  $P \star Q \notin \Delta_0$  if and only if  $P = Q$ .

**Theorem 3.3.** *Let  $X, Y \subseteq \mathbb{P}^2$  be two sets of points with  $|X|, |Y| \geq 3$ ,  $|X \cup Y| \geq 4$  and  $X \cup Y \subseteq L$ , where  $L$  is a line. Then the points of  $X \star Y$  are not collinear if and only if  $L \cap \Delta_0 = \emptyset$ .*

*Proof.* The necessary part follows from Corollary 3.1.

To prove the sufficient part, first we note that, by (1) of Lemma 3.2,  $|\Delta_1 \cap L| = 3$ , and so  $X \cup Y \not\subseteq \Delta_1$ , thus there exists at least one point of  $X$  or  $Y$  not in  $\Delta_1$ .

Suppose that there exists a unique point  $P_1 \in X$  such that  $P_1 \notin \Delta_1$ , hence  $X \setminus \{P_1\} \subseteq L \cap \Delta_1$ .

Then either there exists  $Q_1 \in Y$  with  $Q_1 \neq P_1$  such that  $Q_1 \notin \Delta_1$  or  $Y \setminus \{P_1\} \subseteq L \cap \Delta_1$ .

In the first case for each  $P \in X$  and for each  $Q \in Y$ , with  $P \neq P_1$  and  $Q \neq Q_1$ , we have that  $P \star Q_1 \neq P_1 \star Q_1 \neq P_1 \star Q$ , by (1) of Theorem 2.1.

Since  $P_1 \star L \neq Q_1 \star L$  by (2) of Theorem 2.1, the points  $P \star Q_1, P_1 \star Q_1, P_1 \star Q$  cannot be collinear.

In the second case we can show that the points of  $X \star (Y \setminus \{P_1\})$  are not collinear. In fact, for every  $Q, Q' \in Y \setminus \{P_1\}$  with  $Q \neq Q'$  and for each  $P \in X \setminus \{P_1\}$ , we have  $Q \star L \subseteq H_i$  and  $Q' \star L \subseteq H_j$ , with  $i \neq j$ , since  $L \cap \Delta_0 = \emptyset$ . Then  $P_1 \star Q, P \star Q \in H_i$  and  $P_1 \star Q', P \star Q' \in H_j$ . Now, we have either  $P \neq Q$  or  $P \neq Q'$ , whence, by (2) of Lemma 3.2,  $P \star Q \in \Delta_0$  or  $P \star Q' \in \Delta_0$ . Thus, either  $P_1 \star Q \neq P \star Q \neq P_1 \star Q'$  or  $P_1 \star Q \neq P \star Q' \neq P_1 \star Q'$ , for  $P_1 \star Q, P_1 \star Q' \notin \Delta_0$ . On the other hand,  $P_1 \star Q \neq P_1 \star Q'$  by (2) of Theorem 2.1. Since  $P_1 \star Q, P_1 \star Q' \notin H_i \cap H_j \subset \Delta_0$ , the points  $P_1 \star Q, P \star Q, P_1 \star Q', P \star Q'$  cannot be collinear.

Now suppose that there exist at least  $P_1, P_2 \in X$  and  $Q_1, Q_2 \in Y$ , all distinct, such that either  $P_1, P_2 \notin \Delta_1$  or  $Q_1, Q_2 \notin \Delta_1$ .

We may assume that  $P_1, P_2 \notin \Delta_1$ , the other case being similar.

For every  $Q, Q' \in Y$ , with  $Q \neq Q'$  we have that  $P_1 \star Q \neq P_1 \star Q'$  and  $P_2 \star Q \neq P_2 \star Q'$ , by (1) of Theorem 2.1. Since  $P_1 \star L \neq P_2 \star L$ , by (1) of Theorem 2.1, the points  $P_1 \star Q, P_1 \star Q', P_2 \star Q, P_2 \star Q'$  cannot be all collinear.  $\square$

By Lemma 3.1 of [BCK] we have that if  $L \subset \mathbb{P}^n$  is a linear subspace of dimension  $m$  and  $P$  is a point, then  $P \star L$  is either empty or it is a linear subspace of dimension at most  $m$ . If  $P \notin \Delta_{n-1}$  then  $\dim(P \star L) = m$ . In the following Theorem we give a description of what occurs in the plane in some cases. We will use these technical results later.

**Theorem 3.4.** *Let  $L$  be the line in  $\mathbb{P}^2$  defined by  $a_0x_0 + a_1x_1 + a_2x_2 = 0$ , set  $A = [a_0 : a_1 : a_2] \in \mathbb{P}^2$  and let  $Q = [q_0 : q_1 : q_2]$  be any point. Then:*

- (1) *if  $Q \notin \Delta_1$ , then  $Q \star L$  is the line defined by  $\frac{a_0}{q_0}x_0 + \frac{a_1}{q_1}x_1 + \frac{a_2}{q_2}x_2 = 0$ ;*
- (2) *if  $Q \in \Delta_1 \setminus \Delta_0$  (and so  $Q \in H_j$ , for some  $j \in \{0, 1, 2\}$ ),  $A \in \Delta_i \setminus \Delta_{i-1}$ , ( $i \in \{0, 1, 2\}$ ) and  $Q \star A \in \Delta_{i-1}$ , then  $Q \star L = H_j$ ;*
- (3) *if  $Q \in \Delta_1 \setminus \Delta_0$  (and so  $Q \in H_j$  for some  $j \in \{0, 1, 2\}$ ) and we are not in the hypothesis of (2), then  $Q \star L$  is the point of the intersection of the line  $H_j$  with the line defined by  $a_kq_lx_k + a_lq_kx_l = 0$ , with  $k, l \neq j$ ;*
- (4) *if  $Q \in \Delta_0$  and  $Q \neq A$ , then  $Q \star L = Q$ ;*
- (5) *if  $Q \in \Delta_0$  and  $Q = A$ , then  $Q \star L$  is not defined.*

*Proof.* (1) If  $A \notin \Delta_1$  the result follows immediately from (2) of Theorem 2.1.

Now we suppose  $A \in \Delta_1 \setminus \Delta_0$ , say  $A \in H_2$ , then  $L : a_0x_0 + a_1x_1 = 0$ . Let  $P \in L$ , then  $P = [-\frac{a_1}{a_0}p_1 : p_1 : p_2]$  and  $P \star$

$Q = [-\frac{a_1}{a_0}q_0p_1 : q_1p_1 : q_1p_2]$ , whence  $Q \star L$  is contained in the line  $a_0q_1x_0 + a_1q_0x_1 = 0$ . To see the other inclusion consider  $S = [-\frac{a_1q_0}{a_0q_1}s_1 : s_1 : s_2]$ . Then we have that  $S = Q \star P$ , where  $P = [-\frac{a_1}{a_0q_1}s_1 : \frac{s_1}{q_1} : \frac{s_2}{q_2}]$ . Since  $a_0(-\frac{a_1}{a_0q_1}s_1) + a_1(\frac{s_1}{q_1}) = 0$ , then  $P \in L$ .

The proof is similar if we suppose  $a_0 = 0$  or  $a_1 = 0$ .

Finally suppose  $A \in \Delta_0$ , so that  $L = H_i$ , for some  $i \in \{0, 1, 2\}$ . In this case it is easy to see that  $Q \star L$  is  $H_i$ .

- (2), (3) Since  $Q \in \Delta_1 \setminus \Delta_0$ , without loss of generality, we may assume that  $Q = [0 : q_1 : q_2]$ . First consider the case  $i = 2$ , i.e.  $A \notin \Delta_1$ . In this case necessarily  $Q \star A \in \Delta_1$ . Let  $P \in L$ , then  $P = [-\frac{a_1p_1+a_2p_2}{a_0} : p_1 : p_2]$  and  $P \star Q = [0 : p_1q_1 : p_2q_2]$ , i.e.  $Q \star L = H_0$ , because  $q_1q_2 \neq 0$ .

Now suppose  $i = 1$ , i.e.  $A \in \Delta_1 \setminus \Delta_0$ . In this case we need to distinguish whether  $Q \star A \in \Delta_0$  or not.

If  $Q \star A \in \Delta_0$ , then  $a_0 \neq 0$  and we may assume  $a_2 = 0$ , i.e.  $L : \{a_0x_0 + a_1x_1 = 0\}$ .

Let  $P \in L$ , then  $P = [-\frac{a_1p_1}{a_0} : p_1 : p_2]$  and  $P \star Q = [0 : p_1q_1 : p_2q_2]$ , i.e.  $Q \star L = H_0$ , because  $q_1q_2 \neq 0$ .

If  $Q \star A \in \Delta_1 \setminus \Delta_0$ , then we may assume  $q_0 = a_0 = 0$ , i.e.  $L : \{a_1x_1 + a_2x_2 = 0\}$ . Let  $P \in L$ , then  $P = [p_0 : -\frac{a_2}{a_1}p_2 : p_2]$  and  $Q \star P = [0 : -\frac{a_2}{a_1}p_2q_1 : p_2q_2] = [0 : -\frac{a_2}{a_1}q_1 : q_2]$ , whence  $Q \star L = [0 : -\frac{a_2}{a_1}q_1 : q_2] = H_0 \cap \{a_1q_2x_1 + a_2q_2x_2 = 0\}$ .

Now suppose  $i = 0$ , i.e.  $A \in \Delta_0$ , then  $Q \star A$  can be defined or not.

If  $Q \star A \in \Delta_{-1}$ , then  $L : \{x_0 = 0\} = H_0$  and  $Q \star L = H_0$ .

If  $Q \star A \notin \Delta_{-1}$ , then we may assume  $L : \{x_1 = 0\} = H_1$ . In this case  $Q \star L = \{[0 : 0 : 1]\} = H_0 \cap \{x_1 = 0\}$ .

- (4), (5) They follow immediately from the definition of the Hadamard product.

□

**Corollary 3.5.** *Let  $X$  be a set of collinear points in  $\mathbb{P}^2$  and let  $Q$  be a point. Then  $Q \star X$  is contained in a line.*

**Theorem 3.6.** *Let  $L$  be a line in  $\mathbb{P}^2$  defined by  $a_0x_0 + a_1x_1 + a_2x_2 = 0$ , set  $A = [a_0 : a_1 : a_2]$  and let  $Q = [q_0 : q_1 : q_2]$ ,  $Q' = [q'_0 : q'_1 : q'_2]$  be two distinct points. Suppose  $A, Q, Q' \notin \Delta_0$ , then:*

- (1)  $Q \star L$  and  $Q' \star L$  are lines when:
  - (a)  $A \notin \Delta_1$  and either  $Q \notin \Delta_1$  or  $Q' \notin \Delta_1$ . The two lines are distinct;
  - (b)  $A \notin \Delta_1$ ,  $Q, Q' \in \Delta_1$ . The two lines are distinct if and only if  $Q \star Q' \in \Delta_0$ ;
  - (c)  $A \in \Delta_1$ ,  $Q, Q' \notin \Delta_1$ . The two lines are distinct if and only if  $\det \begin{pmatrix} q_j & q_k \\ q'_j & q'_k \end{pmatrix} \neq 0$  where  $j, k \neq i$  and  $A \in H_i$ ;

- (d)  $A \in \Delta_1$ ,  $Q' \notin \Delta_1$  and  $Q \star A \in \Delta_0$ . The two lines are distinct;
- (e)  $A \in \Delta_1$ ,  $Q \star A \in \Delta_0$  and  $Q' \star A \in \Delta_0$ . The two lines are distinct if and only if  $Q \star Q' \in \Delta_0$ .
- (2)  $Q \star L$  is a point and  $Q' \star L$  is a line when:
  - (a)  $A, Q \in \Delta_1$ ,  $Q' \notin \Delta_1$  and  $Q \star A \notin \Delta_0$ . The point  $Q \star L$  belongs to the line  $Q' \star L$  if and only if  $\det \begin{pmatrix} q_j & q_k \\ q'_j & q'_k \end{pmatrix} = 0$ , where  $j, k \neq i$  and  $A \in H_i$ ;
  - (b)  $A, Q \in \Delta_1$ ,  $Q \star A \notin \Delta_0$  and  $Q' \star A \in \Delta_0$ . The point  $Q \star L$  does not belong to the line  $Q' \star L$ .
- (3)  $Q \star L$  and  $Q' \star L$  are two distinct points when  $A, Q, Q' \in \Delta_1$ ,  $Q \star A \notin \Delta_0 \not\equiv Q' \star A$ .

*Proof.* (1)-(a) If  $Q, Q' \notin \Delta_1$  then  $Q \star L$  and  $Q' \star L$  are two distinct lines by (2) of Theorem 2.1.

If  $Q \in \Delta_1$  and  $Q' \notin \Delta_1$ , then, by (2) of Theorem 3.4,  $Q \star L = H_i$ , for some  $i \in \{0, 1, 2\}$ , which is a line distinct from  $Q' \star L$ .

- (1)-(b) Since  $Q, Q' \in \Delta_1 \setminus \Delta_0$ , then, by (2) of Theorem 3.4,  $Q \star L = H_i$  and  $Q' \star L = H_j$ , for some  $i, j \in \{0, 1, 2\}$ . Clearly, these lines are distinct if and only if  $Q \star Q' \in \Delta_0$ .

In the rest of the proof we have  $A \in \Delta_1 \setminus \Delta_0$  and so, without loss of generality, we may assume  $A = [0 : a_1 : a_2] \in H_0$ , with  $a_1 a_2 \neq 0$ , whence  $L : \{a_1 x_1 + a_2 x_2 = 0\}$ .

- (1)-(c) Since  $Q, Q' \notin \Delta_1$ , then, by (1) of Theorem 3.4,  $Q \star L : \{a_1 q_2 x_1 + a_2 q_1 x_2 = 0\}$  and  $Q' \star L : \{a_1 q'_2 x_1 + a_2 q'_1 x_2 = 0\}$  are lines. Clearly, these lines are distinct if and only if  $\det \begin{pmatrix} q_1 & q_2 \\ q'_1 & q'_2 \end{pmatrix} \neq 0$ .
- (1)-(d) Since  $Q' \notin \Delta_1$ , then  $Q' \star L : \{a_1 q'_2 x_1 + a_2 q'_1 x_2 = 0\}$ , by (1) of Theorem 3.4. Since  $Q \star A \in \Delta_0$ , then necessarily  $Q \in \Delta_1$ , and so, by (2) of Theorem 3.4,  $Q \star L = H_i$ , for some  $i \in \{1, 2\}$ . These two lines are distinct because  $Q' \notin \Delta_1$ .
- (1)-(e) Since  $Q \star A, Q' \star A \in \Delta_0$ , then necessarily  $Q, Q' \in \Delta_1$ , and so by (2) of Theorem 3.4,  $Q \star L = H_i$  and  $Q' \star L = H_j$ , for some  $i \in \{1, 2\}$ . Clearly, these lines are distinct lines if and only if  $Q \star Q' \in \Delta_0$ .
- (2)-(a) Since  $Q \star A \notin \Delta_0$ , then, by (3) of Theorem 3.4,  $Q \star L = [0 : -a_2 q_1 : a_1 q_2]$ , which belongs to the line  $Q' \star L : \{a_1 q'_2 x_1 + a_2 q'_1 x_2 = 0\}$  if and only if  $\det \begin{pmatrix} q_1 & q_2 \\ q'_1 & q'_2 \end{pmatrix} = 0$ .
- (2)-(b) Since  $Q \star A \notin \Delta_0$  and  $Q' \star A \in \Delta_0$ , then, by (2), (3) of Theorem 3.4,  $Q \star L = [0 : -a_2 q_1 : a_1 q_2]$  and  $Q' \star L = H_i$  for some  $i \in \{1, 2\}$ . Clearly, the point does not belong to the line.
- (3) Since  $Q \star A, Q' \star A \notin \Delta_0$  and  $A, Q, Q' \in \Delta_1$ , then, by (3) of Theorem 3.4,  $Q \star L = [0 : -a_2 q_1 : a_1 q_2]$ , and  $Q' \star L = [0 :$



$-a_2q'_1 : a_1q'_2]$ . These two points are distinct because  $Q \neq Q'$  and  $Q, Q' \in \Delta_1$  imply  $\det \begin{pmatrix} q_1 & q_2 \\ q'_1 & q'_2 \end{pmatrix} \neq 0$ .

□

**Remark 3.7.** Let  $L$  be a line in  $\mathbb{P}^2$  such that  $L \cap \Delta_0 = \emptyset$ , and let  $P, Q \in L$ , with  $P \neq Q$ . Then the minors of order two of the matrix  $\begin{pmatrix} P \\ Q \end{pmatrix}$  are all not zero.

**Theorem 3.8.** Let  $L$  be a line in  $\mathbb{P}^2$  defined by  $a_0x_0 + a_1x_1 + a_2x_2 = 0$  and let  $A = [a_0 : a_1 : a_2]$ . Let  $L'$  be a line in  $\mathbb{P}^2$  defined by  $a'_0x_0 + a'_1x_1 + a'_2x_2 = 0$  and such that  $[a'_0 : a'_1 : a'_2] \notin \Delta_1$ . Let  $X' \subseteq L'$  be a set of  $r$  collinear points and suppose  $L' \cap \Delta_1 \subseteq X'$ . Then:

- (1) If  $A \notin \Delta_1$ , then  $X' \star L$  is a set of  $r$  distinct lines.
- (2) If  $A \in \Delta_1 \setminus \Delta_0$ , then  $X' \star L$  is a set of  $r - 1$  distinct lines and a point which does not belong to any line of  $X' \star L$ .
- (3) If  $A \in \Delta_0$ , then  $X' \star L = L$ .

*Proof.* (1) Since  $A \notin \Delta_1$ , then  $\{P' \star L | P' \in X' \setminus \Delta_1\}$  is a set of  $r - 3$  distinct lines. In fact, by (1) of Lemma 3.2,  $|L \cap \Delta_1| = 3$ , and by (2) of Theorem 2.1, each  $P' \star L$  is a line and these lines are all distinct.

By (2) of Theorem 3.4, if  $P' \in X' \cap \Delta_1$ , then  $P' \star L = H_i$  (for some  $i \in \{0, 1, 2\}$ ) and they are distinct by (1)-(b) of Theorem 3.6;

- (2) Since  $A \in \Delta_1 \setminus \Delta_0$ , then  $\{P' \star L | P' \in X' \setminus \Delta_1\}$  is a set of  $r - 3$  distinct lines. In fact, by (1) of Theorem 3.4, each  $P' \star L$  is a line and it is easy to prove that they are all distinct by using Remark 3.7.

Since  $L' \cap \Delta_1 \subseteq X'$ , by (1) of Lemma 3.2 there exists only one point  $P'_1 \in X' \cap \Delta_1$  such that  $P'_1 \star A \notin \Delta_0$ . Thus, by (3) of Theorem 3.4,  $P'_1 \star L$  is a point which does not belong to any of the lines of  $\{P' \star L | P' \in X' \setminus \Delta_1\}$  by (2)-(a) of Theorem 3.6 in view of Remark 3.7. For the remaining two points  $P'_2, P'_3 \in X' \setminus \Delta_1$  we have  $P'_2 \star L = H_i$  and  $P'_3 \star L = H_j$ , for some  $i, j \in \{0, 1, 2\}$ , with  $i \neq j$ .

- (3) Since  $A \in \Delta_0$ , then  $L = H_i$ , for some  $i \in \{0, 1, 2\}$ . By Remark 2.7  $X' \star L \subseteq L$ . On the other hand, from  $L' \cap \Delta_1 \subseteq X'$ ,  $[a'_0 : a'_1 : a'_2] \notin \Delta_1$  and (1) of Lemma 3.2, it follows that there exists a point  $P' \in (X' \cap \Delta_1) \setminus \Delta_0$ , such that  $P' \star A \in \Delta_{-1}$ . The conclusion follows from (2) of Theorem 3.4.

□

**Corollary 3.9.** Let  $L$  be a line in  $\mathbb{P}^2$  defined by  $a_0x_0 + a_1x_1 + a_2x_2 = 0$  and let  $A = [a_0 : a_1 : a_2]$ . Let  $L'$  be a line in  $\mathbb{P}^2$  defined by  $a'_0x_0 + a'_1x_1 +$



$a'_2x_2 = 0$  and such that  $[a'_0 : a'_1 : a'_2] \notin \Delta_1$ . Let  $X' \subseteq L'$  be a set of  $r$  collinear points. Then:

- (1) If  $A \notin \Delta_1$ , then  $X' \star L$  is a set of  $r$  distinct lines;
- (2) If  $A \in \Delta_1 \setminus \Delta_0$ , then  $X' \star L$  is either a set of  $r$  distinct lines or a set of  $r - 1$  distinct lines and a point which does not belong to any line of  $X' \star L$ ;
- (3) If  $A \in \Delta_0$  and  $r \geq 3$ , then  $X' \star L = L$ .

*Proof.* The only difference with the previous Theorem is that we no longer have the hypothesis  $L' \cap \Delta_1 \subseteq X'$ , and so the existence of  $P' \in X' \cap \Delta_1$  such that  $P' \star A \notin \Delta_0$  is not granted, therefore we can obtain  $r$  lines and no extra point.

As for (3), if there exists  $P' \in X' \setminus \Delta_1$ , we are done by (1) of Theorem 3.4. If every  $P' \in X'$  is in  $\Delta_1$ , then, in view of (1) of Lemma 3.2, we have  $r = 3$  and  $L' \cap \Delta_1 = X'$ .  $\square$

**Example 3.10.** Let  $L' \subset \mathbb{P}^2$  be the line of equation  $2x_0 - 3x_1 + 132x_2$  and let  $X' \subset L'$  be the following set of five points (randomly chosen in  $L'$  by Singular)

$$X' = \{[27 : 238 : 5], [12 : 96 : 2], [15 : 142 : 3], [21 : 234 : 5], [33 : 242 : 5]\}$$

After setting  $X' = Y$ , we get that the ideal  $I$  of  $Y$  is generated by  $I[1]$  and  $I[2]$ , where:

$$\begin{aligned} I[1] &= 2*x(0) - 3*x(1) + 132*x(2) \\ I[2] &= 375*x(1)^5 - 89300*x(1)^4*x(2) + 8505840*x(1)^3*x(2)^2 + \\ &\quad - 405077872*x(1)^2*x(2)^3 + 9645291984*x(1)*x(2)^4 - \\ &\quad - 91862394624*x(2)^5 \end{aligned}$$

As  $L$  consider the line  $2x_0 - 3x_1 - 11x_2$ ; clearly we are in the case  $A \notin \Delta_1$ . Computing the Hadamard product  $L \star X'$ , in Singular we get

```
ideal J=2*x(0)-3*x(1)-11*x(2);
ideal YL=HPr(I,J,2);
degree(YL);
// dimension (proj.) = 1
// degree (proj.) = 5
genus(YL);
-4
```

which tell us that  $L \star X'$  is the union of five lines. In particular, looking at the primary decomposition of the ideal  $YL$  we recover the five lines

$$\begin{aligned} 16x_0 - 3x_1 - 528x_2 &= 0 \\ 284x_0 - 45x_1 - 7810x_2 &= 0 \\ 2380x_0 - 405x_1 - 70686x_2 &= 0 \\ 260x_0 - 35x_1 - 6006x_2 &= 0 \\ 220x_0 - 45x_1 - 7986x_2 &= 0 \end{aligned}$$

**Lemma 3.11.** Let  $L$  be the line defined by  $a_0x_0 + a_1x_1 + a_2x_2 = 0$  and  $L'$  defined by  $a'_0x_0 + a'_1x_1 + a'_2x_2 = 0$ , let  $A = [a_0 : a_1 : a_2]$

and  $A' = [a'_0 : a'_1 : a'_2]$  with  $A, A' \notin \Delta_1$ . Let  $P_1, P_2 \in L \setminus \Delta_1$  and  $P'_1, P'_2 \in L' \setminus \Delta_1$  with  $\{P_1, P_2\} \cap \{P'_1, P'_2\} = \emptyset$ . If  $P_1 \star P'_1 = P_2 \star P'_2$ , then either  $P_1 = P_2$  and  $P'_1 = P'_2$  or  $P_i \star A = P'_j \star A'$  for  $i, j \in \{1, 2\}$  with  $i \neq j$ .

*Proof.* If  $P_1 = P_2$ , then because  $P_1, P_2 \notin \Delta_1$  and  $P_1 \star P'_1 = P_2 \star P'_2$ , then  $P'_1 = P'_2$ , by (1) of Theorem 2.1.

Suppose  $P_1 = [p_{10} : p_{11} : p_{12}] \neq P_2 = [p_{20} : p_{21} : p_{22}]$  and  $P'_1 = [p'_{10} : p'_{11} : p'_{12}] \neq P'_2 = [p'_{20} : p'_{21} : p'_{22}]$ . Since  $P'_1 \neq P'_2$ , we have  $P_1 \star P'_2 \neq P_1 \star P'_1 = P_2 \star P'_2$ . Through  $P_1 \star P'_2$  and  $P_1 \star P'_1$  there is only the line  $P_1 \star L'$  defined by  $\frac{a'_0}{p_{10}}x_0 + \frac{a'_1}{p_{11}}x_1 + \frac{a'_2}{p_{12}}x_2 = 0$ , and through  $P_1 \star P'_2$  and  $P_2 \star P'_2$  there is only the line  $P'_2 \star L$  defined by  $\frac{a_0}{p'_{20}}x_0 + \frac{a_1}{p'_{21}}x_1 + \frac{a_2}{p'_{22}}x_2 = 0$ . Since  $P_1 \star P'_1 = P_2 \star P'_2$ , these two lines must coincide, i.e.  $\frac{a'_i}{p_{1i}} = \alpha \frac{a_i}{p'_{2i}}$ , for  $i \in \{0, 1, 2\}$ , which gives  $P_1 \star A = P'_2 \star A'$ .  $\square$

**Theorem 3.12.** *Let  $X$  be a set of  $n$  collinear points, with  $X \subseteq L \setminus \Delta_1$  and  $L$  defined by  $a_0x_0 + a_1x_1 + a_2x_2 = 0$ . Let  $X'$  be a set of  $m$  collinear points, with  $X' \subseteq L' \setminus \Delta_1$  and  $L'$  defined by  $a'_0x_0 + a'_1x_1 + a'_2x_2 = 0$ . Set  $A = [a_0 : a_1 : a_2]$  and  $A' = [a'_0 : a'_1 : a'_2]$ . If  $A, A' \notin \Delta_1$  and  $X \cap X' = \emptyset$ , then,  $X \star X'$  is the  $nm$  element grid  $(X \star L') \cap (X' \star L)$  if and only if  $P \star A \neq P' \star A'$ , for all  $P \in X$  and all  $P' \in X'$ .*

*Proof.* If  $(X \star L') \cap (X' \star L)$  is a grid with  $nm$  elements, then the lines  $\{P \star L' \mid P \in X\}$  and  $\{P' \star L \mid P' \in X'\}$  are all distinct. With the same reasoning of Lemma 3.11, we can prove that  $P \star A \neq P' \star A'$  for all  $P \in X$  and  $P' \in X'$ .

Conversely, since  $P \star A \neq P' \star A'$ , then  $\{P \star L' \mid P \in X\}$  and  $\{P' \star L \mid P' \in X'\}$  are two families of distinct lines by Corollary 3.9. Moreover, since  $P \star A \neq P' \star A'$ , for all  $P \in X$  and  $P' \in X'$ , as in the proof of Lemma 3.11, we obtain that also  $\{P \star L', P' \star L \mid P \in X, P' \in X'\}$  is a family of distinct lines. On the other hand it is easy to check that  $X \star X' = (X \star L') \cap (X' \star L)$ . Now suppose  $X \star X'$  has fewer than  $nm$  elements, then there exist  $P_1, P_2 \in X$  and  $P'_1, P'_2 \in X'$  with  $P_1 \neq P_2$  and  $P'_1 \neq P'_2$  such that  $P_1 \star P'_1 = P_2 \star P'_2$ . By Lemma 3.11 this forces  $P_1 \star A = P'_2 \star A'$  against the hypothesis.  $\square$

**Corollary 3.13.** *Let  $X, Y$  be two disjoint sets of points both contained in the same line  $L$ . Suppose  $X \cap \Delta_1 = \emptyset = Y \cap \Delta_1$  and  $L \cap \Delta_0 = \emptyset$ . If  $|X| = n$  and  $|Y| = m$ , then  $X \star Y$  is the  $nm$  element grid  $(X \star L) \cap (Y \star L)$ .*

*Proof.* Let  $L$  be defined by  $a_0x_0 + a_1x_1 + a_2x_2 = 0$  and let  $A = [a_0 : a_1 : a_2]$ , then, for all  $P \in X$  and all  $P' \in Y$ , we have  $P \star A \neq P' \star A$  by (1) of Theorem 2.1. Now the conclusion follows from Theorem 3.12.  $\square$

**Corollary 3.14.** *Let  $L, L'$  be two generic distinct lines in  $\mathbb{P}^2$ . There is a generic choice of a finite set of points  $X \subseteq L$  for which it is possible a generic choice of a finite set of points  $X' \subseteq L'$  such that  $X \star X'$  is the grid  $(X \star L') \cap (X' \star L)$ .*

*Proof.* Let  $L, L'$  be defined by  $a_0x_0 + a_1x_1 + a_2x_2 = 0$  and  $a'_0x_0 + a'_1x_1 + a'_2x_2 = 0$  respectively and let  $A = [a_0 : a_1 : a_2]$  and  $A' = [a'_0 : a'_1 : a'_2]$ . We may assume  $L \cap \Delta_0 = \emptyset = L' \cap \Delta_0$ , whence  $A, A' \notin \Delta_1$ . Let  $P, Q \in L \setminus \Delta_1$  and  $P' \in L' \setminus \Delta_1$  be distinct points. By (1) of Theorem 2.1  $P \star A \neq Q \star A$ , then either  $P \star A \neq P' \star A'$  or  $Q \star A \neq P' \star A'$ . Suppose  $P \star A \neq P' \star A'$  and consider the  $2 \times 3$  matrix  $M(\lambda, \mu) = \begin{pmatrix} A \\ A' \end{pmatrix} \star \begin{pmatrix} \lambda P + \mu Q \\ P' \end{pmatrix}$  with  $[\lambda : \mu] \in \mathbb{P}^1$ . Then  $M(1, 0)$  has a non-zero  $2 \times 2$  minor. The corresponding minor in  $M(\lambda, \mu)$  is a non-zero linear form  $F(\lambda, \mu)$ . Let  $P_0 \in L$  be the point corresponding to the zero locus of  $F$ . Thus the set  $L \setminus \{P_0\}$  is a non empty open subset  $U$  of  $L$ . Moreover, if  $R \in U$  then  $R \star A \neq P' \star A'$ . Now consider a finite set of points  $X \subseteq U \cap (L \setminus \Delta_1)$ . For any point  $R \in X$ , by the same reasoning as before, we find a non empty open subset  $U'_R$  of  $L'$  such that  $R \star A \neq R' \star A'$  for any point  $R' \in U'_R$ . Set  $U' = \bigcap_{R \in X} U'_R$ . If  $X' \subseteq U' \cap (L' \setminus \Delta_1)$  is a finite set of points then  $P \star A \neq P' \star A'$  for all  $P \in X$  and  $P' \in X'$ .

Now the claim follows from Theorem 3.12.  $\square$

**Remark 3.15.** The grids obtained in Corollaries 3.13 and 3.14 are complete intersections so the Hilbert Functions and even the resolutions are known.

**Example 3.16.** Let  $L$  and  $L'$  be respectively the lines  $3x_0 + x_1 - 30x_2 = 0$  and  $67x_0 - 6x_1 - 110x_2$  (randomly chosen by **Singular**). Consider the sets of points (still randomly chosen by **Singular**), which satisfy the hypotheses of Theorem 3.12,

$$\begin{aligned} X &= \{[6 : 12 : 1], [22 : 54 : 4], [29 : 63 : 5]\} \subset L \\ X' &= Y = \{[22 : 154 : 5], [28 : 221 : 5], [34 : 288 : 5], [18, 146, 3]\} \subset L' \end{aligned}$$

Using the procedure of Section 5 we compute the ideal  $I$  of  $X \star X'$  and then its Hilbert function

```
ideal I=HPr(X,Y,2);
```

```
HF(2,I,0)=1;
HF(2,I,1)=3;
HF(2,I,2)=6;
HF(2,I,3)=9;
HF(2,I,4)=11;
HF(2,I,5)=12;
HF(2,I,6)=12;
```

that is  $H(t) = 12$  for  $t \geq 5$ .

As expected,  $X \star X'$  is a complete intersection.

4. SETS OF COLLINEAR POINTS IN  $\mathbb{P}^3$ 

We keep assuming that the sets of points under consideration have cardinalities at least two.

**Lemma 4.1.** *Let  $L$  be a line in  $\mathbb{P}^3$  such that  $L \cap \Delta_0 = \emptyset$  and let  $H$  be a generic plane through  $L$ . Then  $H \cap \Delta_0 = \emptyset$ . Equivalently, if  $A$  is the point corresponding to  $H$  in the dual space, then  $A \notin \Delta_2$ .*

*Proof.* It is immediate since the planes through  $L$  which contain some coordinate point are finite.  $\square$

**Theorem 4.2.** *Let  $L, L'$  be two lines in  $\mathbb{P}^3$ ,  $L = H \cap K$ ,  $L' = H' \cap K'$ , let  $A, B, A'$  and  $B'$  be the points which correspond to  $H, K, H', K'$  in the dual space and suppose  $A, B, A', B' \notin \Delta_2$ . Let  $X \subseteq L$  and  $X' \subseteq L'$  be two finite sets of points such that  $X \cap \Delta_2 = \emptyset = X' \cap \Delta_2$  and*

$$X \cap X' = \emptyset. \text{ Suppose } \text{rank} \begin{pmatrix} A \\ B \\ A' \\ B' \end{pmatrix} \star \begin{pmatrix} P \\ P \\ P' \\ P' \end{pmatrix} > 2 \text{ for all } P \in X \subset L$$

*and  $P' \in X' \subset L'$ , then,  $X \star X' = (X \star L') \cap (X' \star L)$  and  $|X \star X'| = |X||X'|$ .*

*Proof.* Let  $P = [p_0 : \dots : p_3] \in X$  and  $P' = [p'_0 : \dots : p'_3] \in X'$ , first we show that  $P \star L'$  and  $P' \star L$  are distinct lines. They are lines by Lemma 3.1 of [BCK]. On the other hand, by Corollary 2.3, we have that  $P \star L' = (P \star H') \cap (P \star K')$ , and  $P' \star L = (P' \star H) \cap (P' \star K)$ . If we had  $P \star L' = P' \star L$ , then, after denoting  $\frac{1}{P} = [\frac{1}{p_0} : \dots : \frac{1}{p_3}]$  and

$$\frac{1}{P'} = [\frac{1}{p'_0} : \dots : \frac{1}{p'_3}], \text{ we would have that } \text{rank} \begin{pmatrix} A \\ B \\ A' \\ B' \end{pmatrix} \star \begin{pmatrix} \frac{1}{P'} \\ \frac{1}{P'} \\ \frac{1}{P'} \\ \frac{1}{P'} \end{pmatrix} = 2.$$

But a straightforward computation shows that

$$\text{rank} \begin{pmatrix} A \\ B \\ A' \\ B' \end{pmatrix} \star \begin{pmatrix} \frac{1}{P'} \\ \frac{1}{P'} \\ \frac{1}{P'} \\ \frac{1}{P'} \end{pmatrix} = \text{rank} \begin{pmatrix} A \\ B \\ A' \\ B' \end{pmatrix} \star \begin{pmatrix} P \\ P \\ P' \\ P' \end{pmatrix},$$

in contradiction with the hypothesis.

Now let  $P_1, P_2 \in X$ , we shall show that if  $P_1 \star L' = P_2 \star L'$ , then  $P_1 = P_2$ . In fact, let  $P' \in X'$ , then, we just showed that  $P_1 \star L'$  and  $P' \star L$  are distinct lines, hence  $(P_1 \star L') \cap (P' \star L)$  is the point  $P_1 \star P'$ . Similarly,  $P_2 \star P' = (P_2 \star L') \cap (P' \star L)$ . Therefore  $P_1 \star P' = P_2 \star P'$ , hence  $P_1 = P_2$ , by (1) of Theorem 2.1.

In a similar way we can prove that, for any  $P'_1, P'_2 \in X'$ , if  $P'_1 \star L = P'_2 \star L$ , then  $P'_1 = P'_2$ . Finally, we prove that for any  $P_1, P_2 \in X$  and for any  $P'_1, P'_2 \in X'$ ,  $P_1 \star P'_1 \neq P_2 \star P'_2$  provided  $P_1 \neq P_2$  and  $P'_1 \neq P'_2$ .

Assume, by contradiction, that  $P_1 \star P'_1 = P_2 \star P'_2$  but then we would have  $P_1 \star L' = P'_2 \star L$ .

□

**Remark 4.3.** Observe that, under all hypotheses of Theorem 4.2,

the hypothesis  $\text{rank} \begin{pmatrix} A \\ B \\ A' \\ B' \end{pmatrix} \star \begin{pmatrix} P \\ P \\ P' \\ P' \end{pmatrix} > 2$  forces  $\text{rank} \begin{pmatrix} A \\ B \\ A' \\ B' \end{pmatrix} \star \begin{pmatrix} P \\ P \\ P' \\ P' \end{pmatrix} = 3$ , since  $\text{rank} \begin{pmatrix} A \\ B \\ A' \\ B' \end{pmatrix} \star \begin{pmatrix} P \\ P \\ P' \\ P' \end{pmatrix} = 4$  would imply that  $P' \star L$  and  $P \star L'$  are disjoint, while they meet in  $P \star P'$ .

**Lemma 4.4.** *Hypotheses as in Theorem 4.2. If there exist  $P, Q \in X$  with  $P \neq Q$  such that  $(P \star L') \cap (Q \star L') \neq \emptyset$ , then  $X \star L'$  and  $X' \star L$  are contained in the same plane. Similarly if there exist  $P', Q' \in X'$  with  $P' \neq Q'$  such that  $(P' \star L) \cap (Q' \star L) \neq \emptyset$ .*

*Proof.* We only prove the first statement, the other being similar.

Since  $(P \star L') \cap (Q \star L') \neq \emptyset$ , then they determine a plane  $\Pi$ . Now, let  $P'$  be any point of  $X'$  and consider the line  $P' \star L$ . By the proof of Theorem 4.2, one has

$$(P \star L') \cap (P' \star L) = P \star P' \text{ and } (Q \star L') \cap (P' \star L) = Q \star P'$$

hence  $P \star P', Q \star P'$  are distinct points of  $\Pi$  and thus also the line  $P' \star L$  lies in  $\Pi$ , hence  $X' \star L$  is contained in the plane  $\Pi$ . Now let  $Q'$  be any other point of  $X'$ , then  $(P' \star L) \cap (Q' \star L) \neq \emptyset$  and, from what we have proved,  $P' \star L$  and  $Q' \star L$  both lie in  $\Pi$ . With the same reasoning we have that  $X \star L'$  is contained in the plane determined by  $P' \star L$  and  $Q' \star L$ , which is  $\Pi$ . □

**Corollary 4.5.** *Let  $L, L'$  be two generic distinct lines in  $\mathbb{P}^3$ . There is a generic choice of a finite set of points  $X \subseteq L$  for which it is possible a generic choice of a finite set of points  $X' \subseteq L'$  such that:*

- (1)  $X \star X' = (X \star L') \cap (X' \star L)$  and  $|X \star X'| = |X||X'|$ .
- (2)  $L \star L'$  is an irreducible and non-degenerate quadric, and  $X \star L'$  and  $X' \star L$  are lines of the two different rulings.

*Proof.* (1) We may assume that  $L \cap \Delta_1 = \emptyset = L' \cap \Delta_1$ , so that  $L \cap \Delta_2$  and  $L' \cap \Delta_2$  are finite. By Lemma 4.1 we can write  $L = H \cap K$  and  $L' = H' \cap K'$ , with  $A, B, A', B' \notin \Delta_2$ , where  $A, B, A'$  and  $B'$  are the points which correspond to  $H, K, H', K'$  in the dual space.

If  $\text{rank} \begin{pmatrix} A \\ B \\ A' \\ B' \end{pmatrix} \star \begin{pmatrix} P \\ P \\ P' \\ P' \end{pmatrix} = 2$ , for all  $P \in L \setminus \Delta_2$  and  $P' \in L' \setminus \Delta_2$ , then  $P \star L' = P' \star L$  is a line, say  $L''$ , for all  $P \in L \setminus \Delta_2$  and  $P' \in L' \setminus \Delta_2$ . Thus

$$\begin{aligned} L \star L' &= \overline{\bigcup_{P \in L} \{P \star L'\}} = \overline{\left( \bigcup_{P \in L \setminus \Delta_2} \{P \star L'\} \right) \bigcup \left( \bigcup_{P \in L \cap \Delta_2} \{P \star L'\} \right)} = \\ &= L'' \bigcup \left( \bigcup_{P \in L \cap \Delta_2} \{P \star L'\} \right), \end{aligned}$$

which is a union of a line and a finite number of linear spaces of dimension less than or equal to 1. This contradicts Theorem 6.8 of [BCK] in view of Remark 6.9 of [BCK].

Hence there exist  $P \in L \setminus \Delta_2$  and  $P' \in L' \setminus \Delta_2$  such that

$$\text{rank} \begin{pmatrix} A \\ B \\ A' \\ B' \end{pmatrix} \star \begin{pmatrix} P \\ P \\ P' \\ P' \end{pmatrix} = 3.$$

Consider a point  $Q \in L$  and the  $4 \times 4$  matrix

$$M(\lambda, \mu) = \begin{pmatrix} A \\ B \\ A' \\ B' \end{pmatrix} \star \begin{pmatrix} \lambda P + \mu Q \\ \lambda P + \mu Q \\ P' \\ P' \end{pmatrix}$$

with  $[\lambda : \mu] \in \mathbb{P}^1$ . Now we get the conclusion by mimicking the proof of Corollary 3.14 and by applying Theorem 4.2.

- (2) Since  $L$  and  $L'$  are generic, by Theorem 6.8 of [BCK],  $L \star L'$  is a quadric, in fact an irreducible one, as noticed right after Remark 2.5 of [BCK]. Since the quadric is irreducible, then

$$(P' \star L) \cap (Q' \star L) = \emptyset \quad \forall P', Q' \in X'$$

and similarly

$$(P \star L') \cap (Q \star L') = \emptyset \quad \forall P, Q \in X.$$

In fact, suppose  $P' \star L$  and  $Q' \star L$  intersect in a point, then, by Lemma 4.4, the lines  $P' \star L$ ,  $Q' \star L$  and  $P \star L'$  are all distinct and lie in the same plane. But then  $L \star L'$  would be reducible.

On the other hand  $P \star L'$  and  $P' \star L$  intersect in  $P \star P'$  for all  $P \in X$  and for all  $P' \in X'$ .

Therefore  $L \star L'$  is also non degenerate.

□

**Remark 4.6.** If both  $|X|$  and  $|X'|$  are strictly greater than 2, then we have at least three skew lines each with at least three points of  $X \star X'$  and this is enough to prove that  $L \star L'$  is the unique quadric through  $X \star X'$ . It would be interesting to understand the geometry of  $X \star X'$  on such a quadric.

**Example 4.7.** In this example we compute the ideal of  $X \star X'$  and its Hilbert function, where  $X$  and  $X'$  are two sets of collinear points satisfying the hypotheses of Corollary 4.5.

Let  $H, K$  be the planes defined by  $x_0 - x_1 + x_2 + 2x_3 = 0$  and  $x_0 + 2x_1 - x_2 + x_3 = 0$  and let  $H', K'$  be the planes defined by  $x_0 + 2x_1 - 2x_2 + x_3 = 0$  and  $2x_0 + 2x_1 + x_2 - 4x_3 = 0$ . Let  $L = H \cap K$  and  $L' = H' \cap K'$ . Choose  $X \subset L$  and  $X' \subset L'$  where

$$X = \{[-2 : 1 : 1 : 1], [-1 : -1 : -2 : 1], [-2 : 3 : 4 : 1]\}$$

and

$$X' = \{[-1 : 2 : 2 : 1], [11 : -8 : -2 : 1], [-7 : 7 : 4 : 1]\}.$$

By computing the ideal of  $X \star X'$  with *Singular*, we obtain ideal  $I = \text{HPr}(X, X', 3)$

$$\begin{aligned} I[1] = & -3/31x(0)^2 + 41/62x(0)x(1) - 15/31x(1)^2 - 169/186x(0)x(2) + \\ & + 59/62x(1)x(2) - 21/62x(2)^2 - 59/62x(0)x(3) + \\ & + 845/186x(1)x(3) - 287/124x(2)x(3) - 105/62x(3)^2 \\ I[2] = & -3/31x(0)x(1)^2 + 18/31x(1)^3 + 10/31x(0)x(1)x(2) - \\ & - 87/31x(1)^2x(2) - 25/93x(0)x(2)^2 + 140/31x(1)x(2)^2 - \\ & - 75/31x(2)^3 - 24/31x(0)x(1)x(3) + 137/31x(1)^2x(3) + \\ & + 35/31x(0)x(2)x(3) - 2261/186x(1)x(2)x(3) + \\ & + 505/62x(2)^2x(3) + 51/31x(0)x(3)^2 - 362/31x(1)x(3)^2 + \\ & + 1443/62x(2)x(3)^2 - 873/31x(3)^3 \\ I[3] = & 6/31x(0)x(1)x(2) - 36/31x(1)^2x(2) - 10/31x(0)x(2)^2 + \\ & + 114/31x(1)x(2)^2 - 90/31x(2)^3 + 42/31x(0)x(2)x(3) - \\ & - 238/31x(1)x(2)x(3) + 303/31x(2)^2x(3) - 1152/155x(0)x(3)^2 + \\ & + 192/31x(1)x(3)^2 + 3762/155x(2)x(3)^2 - 1728/31x(3)^3 \\ I[4] = & 6/31x(0)x(2)^2 - 36/31x(1)x(2)^2 + 54/31x(2)^3 + \\ & + 3456/775x(0)x(1)x(3) - 576/155x(1)^2x(3) - \\ & - 1584/155x(0)x(2)x(3) + 7944/775x(1)x(2)x(3) - \\ & - 1476/155x(2)^2x(3) + 3456/775x(0)x(3)^2 + \\ & + 4608/155x(1)x(3)^2 - 28296/775x(2)x(3)^2 + \\ & + 5184/155x(3)^3 \\ I[5] = & -6/5x(1)^3 + 6x(1)^2x(2) - 10x(1)x(2)^2 + 50/9x(2)^3 + \\ & + 336/155x(0)x(1)x(3) - 1954/155x(1)^2x(3) - \\ & - 112/31x(0)x(2)x(3) + 15742/465x(1)x(2)x(3) - \\ & - 5972/279x(2)^2x(3) - 428/155x(0)x(3)^2 + \\ & + 13652/465x(1)x(3)^2 - 280218/1395x(2)x(3)^2 + \\ & + 91204/1395x(3)^3 \\ I[6] = & -x(1)^2x(2) + 10/3x(1)x(2)^2 - 25/9x(2)^3 + \\ & + 56/31x(0)x(2)x(3) - 884/93x(1)x(2)x(3) + \end{aligned}$$



$$\begin{aligned}
& +2986/279x(2)^2x(3) - 3956/465x(0)x(3)^2 + \\
& +392/31x(1)x(3)^2 + 28958/1395x(2)x(3)^2 - \\
& -16252/279x(3)^3 \\
I[7] = & -x(1)x(2)^2 + 5/3x(2)^3 + 3956/775x(0)x(1)x(3) - \\
& -1176/155x(1)^2x(3) - 5102/465x(0)x(2)x(3) + \\
& +15944/775x(1)x(2)x(3) - 6703/465x(2)^2x(3) + \\
& +3956/775x(0)x(3)^2 + 12724/465x(1)x(3)^2 - \\
& -88388/2325x(2)x(3)^2 + 16252/465x(3)^3 \\
I[8] = & -1/4x(2)^3 - 1278/775x(0)x(1)x(3) - 252/155x(1)^2x(3) + \\
& +687/155x(0)x(2)x(3) + 3228/775x(1)x(2)x(3) - \\
& -172/155x(2)^2x(3) - 4518/775x(0)x(3)^2 - 234/155x(1)x(3)^2 \\
& + 988/775x(2)x(3)^2 - 422/155x(3)^3
\end{aligned}$$

whose Hilbert function is given by

$$\begin{aligned}
HF(3, I, 0) &= 1 \\
HF(3, I, 1) &= 4 \\
HF(3, I, 2) &= 9 \\
HF(3, I, 3) &= 9
\end{aligned}$$

that is  $H(t) = 9$  for  $t \geq 2$ .

The example above shows that the finite set  $X \star X'$  in Corollary 4.5, in general, is not a complete intersection. However we are able to compute its Hilbert function in the case  $|X| = |X'|$  and this allows us to prove that  $X \star X'$  is never a complete intersection as long as  $m > 1$  (obviously it is for  $m = 1$ ).

**Theorem 4.8.** *Hypotheses as in Corollary 4.5. Also suppose  $|X| = |X'| = m$ , then  $\tau_{X \star X'} = m - 1$  and  $HF_{X \star X'} = HF_X HF_{X'}$ .*

*Proof.* For  $m = 2$  the four points of  $X \star X'$  cannot be coplanar since they belong to the two skew lines of  $X \star L'$  and so  $HF_{X \star X'}(t) = 4$ , for all  $t \geq 1$ .

For  $m \geq 3$ , by intersection theory we have that the quadric  $L \star L'$  (which is the unique quadric through  $X \star X'$  by Remark 4.6) is a fixed component of  $I_{X \star X'}$  in each degree  $2 \leq t < m$ . Then, for  $0 \leq t < m$  we have

$$HF_{X \star X'}(t) = \binom{t+3}{3} - \binom{t+1}{3} = (t+1)^2$$

which equals  $HF_X(t)HF_{X'}(t)$  since  $HF_X(t) = HF_{X'}(t) = t+1$  for  $t < m$ .

In particular,  $HF_{X \star X'}(m-1) = m^2 = |X||X'|$ , hence  $\tau_{X \star X'} = m-1$  and, for all  $t \geq m-1$ ,  $HF_{X \star X'}(t) = m^2 = HF_X(t)HF_{X'}(t)$ .  $\square$

Obviously Theorem 4.8 works also for  $m = 1$ .

**Remark 4.9.** If  $X$  is a finite set of projective points we set

$$h_X = (HF_X(0), \dots, HF_X(\tau_X)).$$

With this notation we can rephrase Theorem 4.8 as

$$h_{X \star X'} = h_X \star h_{X'}.$$

The following example shows that we may still have  $HF_{X \star X'} = HF_X HF_{X'}$  even when  $|X| \neq |X'|$ . It may be worth to investigate if this is always the case, under the hypotheses of Corollary 4.5 (see Example 4.12).

**Example 4.10.** Let  $H, K$  be the planes defined by  $11x_1 - 14x_2 - 2x_3$  and  $22x_0 - 25x_2 - 13x_3$  and let  $H', K'$  be the planes defined by  $21x_1 - 2x_2 - 11x_3$  and  $7x_0 - 6x_2 + 2x_3$ . Let  $L = H \cap K$  and  $L' = H' \cap K'$ . Choose  $X \subset L$  and  $X' \subset L'$  where

$$X = \{[4 : 4 : 3 : 1], [7 : 4 : 2 : 8], [11 : 8 : 5 : 9]\}$$

and

$$X' = \{[2 : 3 : 4 : 5], [6 : 4 : 9 : 6], [18 : 17 : 30 : 27], [94 : 76 : 149 : 118]\}.$$

Let  $I, J, K$  be respectively the ideals of  $X, X'$  and  $X \star X'$ . By Singular we obtain

HF(3, I, 0)=1	HF(3, J, 0)=1	HF(3, K, 0)=1
HF(3, I, 1)=2	HF(3, J, 1)=2	HF(3, K, 1)=4
HF(3, I, 2)=3	HF(3, J, 2)=3	HF(3, K, 2)=9
HF(3, I, 3)=3	HF(3, J, 3)=4	HF(3, K, 3)=12
HF(3, I, 4)=3	HF(3, J, 4)=4	HF(3, K, 4)=12

**Corollary 4.11.** *Hypotheses as in Corollary 4.5 and  $|X| = |X'| = m \geq 2$ . Then  $X \star X'$  is not a complete intersection.*

*Proof.* First assume  $m = 2$ . Then  $\dim_{\mathbb{K}}(I_{X \star X'})_t = \begin{cases} 0 & t = 0, 1 \\ 6 & t = 2 \end{cases}$ .

Thus a minimal system of generators of  $I_{X \star X'}$  contains at least six quadrics and so  $X \star X'$  cannot be complete intersection.

Now assume  $m \geq 3$ . From Remark 4.6 we know that

$$\dim_{\mathbb{K}}(I_{X \star X'})_t = \begin{cases} 0 & t = 0, 1 \\ 1 & t = 2 \end{cases},$$

and so  $\dim_{\mathbb{K}}(I_{X \star X'})_t \geq \binom{t+1}{3}$ ,  $\forall t \geq 2$ . As in the proof of Theorem 4.8 we have that the quadric  $L \star L'$  is a fixed component of  $I_{X \star X'}$  in each degree  $2 \leq t < m$ , and so we need  $\binom{m+3}{3} - m^2 - \binom{m+1}{3} = 2m + 1$  generators of degree  $m$ . Thus a minimal system of generators of  $I_{X \star X'}$  consists of  $2m + 2 > 3$  forms and so  $X \star X'$  cannot be complete intersection.  $\square$

If we drop some of the assumptions of Corollary 4.5 several behaviours may occur, as the following examples show.

**Example 4.12.** Let  $H, K$  be the planes defined by  $x_1 - x_3$  and  $14x_0 - 27x_2 + 10x_3$  and let  $H', K'$  be the planes defined by  $9x_1 + 5x_2 - 11x_3$  and  $x_2 - x_3$ . Let  $L = H \cap K$  and  $L' = H' \cap K'$ . Note that this time  $L \cap \Delta_1 \neq \emptyset$  and  $L' \cap \Delta_1 \neq \emptyset$ . Choose  $X \subset L$  and  $X' \subset L'$  where

$$X = \left\{ \begin{array}{l} [1 : 4 : 2 : 4], [8 : 5 : 6 : 5], [37 : 40 : 34 : 40], \\ [9 : 9 : 8 : 9], [65 : 98 : 70 : 98] \end{array} \right\}$$

and

$$X' = \left\{ \begin{array}{l} [2 : 5 : 2 : 5], [3 : 2 : 3 : 3], [24 : 27 : 24 : 33], \\ [13 : 16 : 13 : 19], [130 : 127 : 130 : 163] \end{array} \right\}.$$

Let  $I, J, K$  be respectively the ideals of  $X, X'$  and  $X \star X'$ . By **Singular** we obtain

HF(3, I, 0)=1	HF(3, J, 0)=1	HF(3, K, 0)=1
HF(3, I, 1)=2	HF(3, J, 1)=2	HF(3, K, 1)=3
HF(3, I, 2)=3	HF(3, J, 2)=3	HF(3, K, 2)=6
HF(3, I, 3)=4	HF(3, J, 3)=4	HF(3, K, 3)=10
HF(3, I, 4)=5	HF(3, J, 4)=5	HF(3, K, 4)=15
HF(3, I, 5)=5	HF(3, J, 5)=5	HF(3, K, 4)=19
HF(3, I, 6)=5	HF(3, J, 6)=5	HF(3, K, 6)=22
HF(3, I, 7)=5	HF(3, J, 7)=5	HF(3, K, 7)=24
HF(3, I, 8)=5	HF(3, J, 8)=5	HF(3, K, 8)=25

Notice that, in this case, the Hilbert function of  $X \star X'$  is not the product of the Hilbert functions of  $X$  and  $X'$ .

As a matter of fact, looking at the ideal of  $X \star X'$ , we can notice that the first generator is

$$K[1]=14*x(0)-18*x(1)-27*x(2)+22*x(3)$$

that is,  $X \star X'$  is a planar set of points. Moreover the first difference of its Hilbert function is  $(1, 2, 3, 4, 5, 4, 3, 2, 1)$  showing that  $X \star X'$  is a complete intersection.

The following two examples show that  $L \star L'$  can be a quadric (necessarily irreducible) also under the condition that  $L \cap \Delta_1 \neq \emptyset$  or  $L \cap \Delta_1 \neq \emptyset \neq L' \cap \Delta_1$ . In both examples  $X \star X'$  is not a complete intersection.

**Example 4.13.** In this example we compute the ideal of  $L \star L'$  and the ideal of  $X \star X'$  with its Hilbert function, where  $X$  and  $X'$  are two sets of collinear points satisfying the hypotheses of Theorem 4.2,  $L \cap \Delta_1 \neq \emptyset$  and  $L' \cap \Delta_1 = \emptyset$ .

Let  $H, K$  be the planes defined by  $x_0 + 2x_1 + x_2 + x_3 = 0$  and  $x_0 + x_1 + x_2 - 3x_3 = 0$  and let  $H', K'$  be the planes defined by  $x_0 + 2x_1 - 2x_2 + x_3 = 0$  and  $2x_0 + 2x_1 + x_2 - 4x_3 = 0$ . Let  $L = H \cap K$  and  $L' = H' \cap K'$ . Choose  $X \subset L$  and  $X' \subset L'$  where

$$X = \{[4 : -4 : 3 : 1], [6 : -4 : 1 : 1], [5 : -4 : 2 : 1]\}$$

and

$$X' = \{[-1 : 2 : 2 : 1], [11 : -8 : -2 : 1], [-7 : 7 : 4 : 1]\}.$$

By computing the ideal of  $L \star L'$  and the ideal of  $X \star X'$  with `Singular`, we obtain

```
ideal J=HPr(L,L',3)
J[1]=1/5xy-21/50y^2-3/5yz-12/5xw+77/25yw-14/5zw+588/25w^2

ideal I=HPr(X,X',3)
I[1]=1/5x(0)x(1)-21/50x(1)^2-3/5x(1)x(2)-12/5x(0)x(3)+
      +77/25x(1)x(3)-14/5x(2)x(3)+588/25x(3)^2
I[2]=1/5x(0)^3-9261/5000x(1)^3-9/5x(0)^2x(2)+
      +27/5x(0)x(2)^2-27/5x(2)^3-15x(0)^2x(3)+
      +25137/625x(1)^2x(3)+27x(0)x(2)x(3)+
      +54x(2)^2x(3)+370x(0)x(3)^2-350763/1250x(1)x(3)^2-
      -165x(2)x(3)^2-1389774/625x(3)^3
I[3]=-x(0)^2x(2)+441/100x(1)^2x(2)+6x(0)x(2)^2-9x(2)^3+
      +40x(0)x(2)x(3)-1071/25x(1)x(2)x(3)+90x(2)^2x(3)-
      -15x(1)x(3)^2-14274/25x(2)x(3)^2+180x(3)^3
I[4]=-x(0)x(2)^2+21/10x(1)x(2)^2+3x(2)^3-3/10x(1)^2x(3)-
      -11/2x(1)x(2)x(3)-101/5x(2)^2x(3)+36/5x(1)x(3)^2+
      +66x(2)x(3)^2-216/5x(3)^3
I[5]=1/10x(1)^3+2/5x(1)^2x(3)-464/5x(1)x(3)^2-3584/5x(3)^3
I[6]=-x(1)^2x(2)-16x(1)x(2)x(3)+320x(0)x(3)^2-672x(1)x(3)^2-
      -224x(2)x(3)^2-3136x(3)^3
I[7]=-x(1)x(2)^2+32/5x(0)^2x(3)-3528/125x(1)^2x(3)-
      -152/5x(0)x(2)x(3)-84/5x(1)x(2)x(3)+28/5x(2)^2x(3)-
      -448x(0)x(3)^2+84672/125x(1)x(3)^2-392/5x(2)x(3)^2+
      +471968/125x(3)^3
I[8]=-25/16x(2)^3-6x(0)^2x(3)+5367/200x(1)^2x(3)+
      +57/2x(0)x(2)x(3)+181/8x(1)x(2)x(3)+6x(2)^2x(3)+
      +365x(0)x(3)^2-53229/100x(1)x(3)^2+349/4x(2)x(3)^2-
      -76294/25x(3)^3
```

whose Hilbert function is given by

```
HF(3,I,0)=1
HF(3,I,1)=4
HF(3,I,2)=9
HF(3,I,3)=9
```

that is  $H(t) = 9$  for  $t \geq 2$ .

**Example 4.14.** In this example we compute the ideal of  $L \star L'$  and the ideal of  $X \star X'$  with its Hilbert function, where  $X$  and  $X'$  are two sets of collinear points satisfying the hypotheses of Theorem 4.2 and  $L \cap \Delta_1 \neq \emptyset \neq L' \cap \Delta_1$ .

Let  $H, K$  be the planes defined by  $x_0 + 2x_1 + x_2 + x_3 = 0$  and  $x_0 + x_1 + x_2 - 3x_3 = 0$  and let  $H', K'$  be the planes defined by  $x_0 + x_1 - 2x_2 + x_3 = 0$  and  $x_0 + x_1 + x_2 - 4x_3 = 0$ . Let  $L = H \cap K$  and  $L' = H' \cap K'$ . Choose  $X \subset L$  and  $X' \subset L'$  where

$$X = \{[4 : -4 : 3 : 1], [6 : -4 : 1 : 1], [5 : -4 : 2 : 1]\}$$

and

$$X' = \{[1 : -1 : \frac{5}{3} : 1], [2 : -2 : \frac{5}{3} : 1], [3 : -3 : \frac{5}{3} : 1]\}.$$

By computing the ideal of  $L \star L'$  and the ideal of  $X \star X'$  with `Singular`, we obtain

```
ideal J=HPr(L,L',3);
J[1]=-3/5x(1)x(2)-4x(0)x(3)+7x(1)x(3)-28/5x(2)x(3)+196/3x(3)^2

ideal I=HPr(x(0),x(0)',3);
I[1]=-3/5x(1)x(2)-4x(0)x(3)+7x(1)x(3)
I[2]=-3/5x(2)^3+6x(2)^2x(3)-55/3x(2)x(3)^2+50/3x(3)^3
I[3]=-x(0)x(2)^2-5/3x(0)x(2)x(3)-50x(0)x(3)^2+250/3x(1)x(3)^2
I[4]=-x(0)^2x(2)-40/3x(0)^2x(3)+185/6x(0)x(1)x(3)-
-25/2x(1)^2x(3)
I[5]=1/4x(1)^3-6x(1)^2x(3)+44x(1)x(3)^2-96x(3)^3
I[6]=-x(0)x(1)^2+24x(0)x(1)x(3)-176x(0)x(3)^2-288/5x(2)x(3)^2+
+672x(3)^3
I[7]=-x(0)^2x(1)+24x(0)^2x(3)+132/5x(0)x(2)x(3)+
+216/25x(2)^2x(3)-308x(0)x(3)^2-1008/5x(2)x(3)^2+
+1176x(3)^3
I[8]=-x(0)^3+90x(0)^2x(3)-111x(0)x(1)x(3)+45x(1)^2x(3)+
+99x(0)x(2)x(3)+162/5x(2)^2x(3)-341x(0)x(3)^2-
-330x(1)x(3)^2-2448/5x(2)x(3)^2+2022x(3)^3
```

whose Hilbert function is given by

```
HF(3,I,0)=1
HF(3,I,1)=4
HF(3,I,2)=9
HF(3,I,3)=9
```

that is  $H(t) = 9$  for  $t \geq 2$ .

## 5. COMPUTING HADAMARD PRODUCTS IN SINGULAR

Given the ideals  $I$  and  $J$  of respectively varieties  $X$  and  $Y$  in  $\mathbb{P}^n$ , the computation of the ideal of  $X \star Y$  may be achieved with a saturation and elimination as follows:

- Work in the ring  $\mathbb{C}[y_{10}, \dots, y_{1n}, y_{20}, \dots, y_{2n}, x_0, \dots, x_n]$ .
- Form the ideal  $I(y_{1i}) + J(y_{2i}) + \langle x_0 - y_{10}y_{20}, \dots, x_n - y_{1n}y_{2n} \rangle$ .
- Saturate with respect to the product  $x_0 \cdots x_n$ .
- Eliminate the  $2n + 2$  variables  $y_{i0}, \dots, y_{in}$ .

For completeness, we show here a procedure in **Singular** which performs the previous steps and that we used to compute the examples in the paper.

```

LIB "ncalg.lib";
LIB "poly.lib";
LIB "rootsmr.lib";
LIB "elim.lib";

proc HPr(ideal I1, ideal I2, int n) /* where n+1 is the number of variables */
{
    ring RH=0,(y(1..2)(0..n),x(0..n)),dp;
    int i;
    ideal T1;
    ideal T2;
    poly elle1;
    poly elle2;
    poly elle3=1;
    map f1;
    map f2;

    T1=y(1)(0);
    for (i=1; i<=n; i=i+1)
    {
        elle1=y(1)(i);
        T1=T1+elle1;
    }
    f1=r,T1;
    ideal H1=f1(I1);

    T2=y(2)(0);
    for (i=1; i<=n; i=i+1)
    {
        elle2=y(2)(i);
        T2=T2+elle2;
    }
    f2=r,T2;
    ideal H2=f2(I2);

    int j;
    ideal H=0;
    for (j=0; j<=n; j=j+1)
    {
        H=H+ideal(x(j)-y(1)(j)*y(2)(j));
        elle3=elle3*x(j);
    }

    H=H+H1+H2;
    ideal Ksat=elle3;
    ideal HH=sat(H,Ksat)[1];

    ideal HHH=elim(H,1..2*(n+1));

    setring r;
    ideal HFin=imap(RH,HHH);

    return(HFin);
}

```

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